Lebesgue points of functions from Sobolev classes on metric measure spaces in the limiting case

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Będlewo, 19 September, 2017

Lebesgue points

Theorem

Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p \geqslant 1$ — summable function. Following property takes place a.e.

$$f(x) = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, dy$$

- What if function is non-summable $(0 < p < 1)$?
- What can we say about Lebesgue points, when f is more regular (for example, belongs to certain function space)?

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Lebesgue theorem: $\mu(\Lambda(f)) = 0$ for any function $f \in L^1_{\mathrm{loc}}(X)$.
$\Lambda(f)$ is an exceptional set — a «small» set for which measure appears to be a «rough» instrument in most applications.

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Approaches to measure size of exceptional set:

- Measure
- Hausdorff measure, content and dimension
- Capacities, generated by function spaces

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(X, d, μ), where d — metric, μ — σ-finite Borel measure

We assume that doubling condition holds

$$\exists\, a_\mu > 0 \quad \mu(B(x, 2r)) \leqslant a_\mu \mu(B(x, r)), \quad x \in X, \quad r > 0.$$

Doubling condition has a quantitative form

$$\exists\, \gamma > 0 \quad \mu(B(x, R) \lesssim \left(\frac{R}{r}\right)^\gamma \mu(B(x, r)), \quad x \in X,\ 0 < r \leqslant R.$$

- One can take $\gamma = \log_2 a_\mu$
- γ plays the role of the dimension of the space

Spaces of homogeneous type

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Sobolev classes on spaces of homogeneous type

Let f be a measurable function on a metric space X, $\alpha > 0$. $g \in D_\alpha[f]$, if there exists a set $E \subset X$, $\mu(E) = 0$ and

$$|f(x) - f(y)| \leqslant [d(x,y)]^\alpha [g(x) + g(y)], \quad x, y \notin E.$$

$$W^p_\alpha(X) = \{f : L^p(X) \cap D_\alpha[f] \neq \varnothing\} \cap L^p(X)$$

$$\|f\|_{W^p_\alpha} = \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)} \,.$$

$W^p_1(\mathbb{R}^n)$ coincides with usual Sobolev space.

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Hausdorff content, measure and dimension

Let $h : (0,1] \to (0,1]$ be an increasing function, $h(+0) = 0$.
Hausdorff content

$$H_R^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(r_i) : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < R \right\}.$$

Hausdorff measure

$$H^h(E) = \lim_{R \to +0} H_R^h(E).$$

Classical case $h(r) = r^s$.
Hausdorff dimension

$$\dim_H(E) = \inf \{ s : H^s(E) = 0 \}.$$

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Capacities

Classes W_α^p generate capacities

$$\mathrm{Cap}_{\alpha,p}(E) = \inf\left\{ ||f||^p_{W^p_\alpha} : f \in W^p_\alpha, f \geqslant 1 \text{ in neighbourhood of E} \subset X \right\}$$

Capacity and measure

Theorem (Kinnunen, Martio, 1996)

Let $0 < \alpha \leqslant 1$, $x_0 \in X$, *then*
1) for any $E \subset X$

$$\mu(E) \leqslant \mathrm{Cap}_{\alpha,p}(E),$$

2) for $0 < r \leqslant 1$

$$\mathrm{Cap}_{\alpha,p}\big(B(x_0,r)\big) \leqslant c r^{-\alpha p} \mu\big(B(x_0,r)\big).$$

Approximation by constants

Let $B \subset X$ be a ball and $f \in L^p(B)$, $p > 0$. Then there exists the number $I_B^{(p)} f \in \mathbb{R}$, such that

$$\inf_{c \in \mathbb{R}} \fint_B |f(y) - c|^p \, d\mu(y) = \fint_B |f(y) - I_B^{(p)} f|^p \, d\mu(y).$$

If $f \in L^p_{\text{loc}}(X)$, then for almost all points $x \in X$ there exists the limit

$$\lim_{r \to +0} I_{B(x,r)}^{(p)} = f^*(x).$$

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Lebesgue point for W_α^p, $p \geqslant 0$

Theorem

Let $0 < \alpha \leqslant 1$, $0 \leqslant p < \gamma/\alpha$ *and* $f \in W_\alpha^p(X)$. *Then there exists a set* $E \subset X$ *such that for any* $x \in X \setminus E$ *the limit*

$$\lim_{r \to +0} I_{B(x,r)}^{(p)} f \, d\mu = f^*(x)$$

exists. Besides,

$$\lim_{r \to +0} \fint_{B(x,r)} |f - f^*(x)|^q \, d\mu = 0, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}.$$

Lebesgue point for W_α^p, $p \geqslant 0$

Following statements take place

$$\mathrm{Cap}_{\alpha,p}(E) = 0$$

$$\dim_{\mathbb{H}}(E) \leqslant \gamma - \alpha p$$

One can take more general outer measure ν satisfying the condition

$$\nu(B) \lesssim r^{-\alpha p}\mu(B).$$

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Lebesgue points for critical case $\gamma = \alpha p$

What can we expect in the critical case $\gamma = \alpha p$?

What is the optimal growth of function ϕ?

$$\lim_{r \to +0} \fint_{B(x,r)} \phi(|f - f_B|)\, d\mu = 0,$$

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Sobolev embeddings

Sobolev embedding theorem says that if $\gamma > \alpha p$, then

$$W_\alpha^p(X) \subset L_{\mathrm{loc}}^q(X), \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$$

If $\gamma = \alpha p$, q tends to infinity. But the embedding

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Poincaré inequality

Sobolev embedding is closely connected to Poincaré inequality

$$\left(\fint_B |f - f_B|^q \right)^{1/q} \leqslant c_q r_B^\alpha \left(\fint_{\sigma B} g^p \right)^{1/p}.$$

When q tends to infinity, c_q blows up.

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Sobolev embedding in critical case

If $\gamma = p > 1$, $\alpha = 1$ and X is connected

$$W_p^1(X) \subset L^{\phi}_{\mathrm{loc}}(X), \;\; \phi(t) = e^{t^{p/(p-1)}} - 1.$$

$X = \mathbb{R}^n$, Trudinger, 1967
Hajłasz–Koskela, 1998
Makmanus–Pérez, 2002
The idea of the proof is to develop an exponent into power series and analyze the behavior of constant c_q from Poincaré inequality. It appears that

$$c_q \asymp q^{(n-1)/n}.$$

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Theorem

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$$\lim_{r \to +0} f_B = f^*(x)$$

exists. Besides, for any constant $c > 0$ and any $0 < \beta \leqslant 1$

$$\lim_{r \to +0} \fint_{B(x,r)} \left[e^{c|f - f_B|^\beta} - 1 \right] d\mu = 0.$$

One can take $I^{(p)}_{B(x,r)} f$ instead of f_B.

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$$\mathrm{Cap}_{\alpha,p}(E) = 0$$

$$H^{\delta}(E) = 0 \text{ for any } \delta > 0, \ \ \dim_{\mathbb{H}}(E) = 0$$

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In case $p > 1$ we can assume even more in previous theorem:

$$\lim_{r \to +0} \fint_{B(x,r)} \left[e^{c\left[\log \frac{1}{r}\right]^{p-1} |f - f_B|} - 1 \right] d\mu = 0.$$

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Thank you for your attention!