Generalized nonlinear heat and Navier-Stokes equations in Besov and Triebel-Lizorkin spaces

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1. Preliminaries
   - Generalized nonlinear heat equation
   - Function spaces
   - Procedure
   - Wavelets and molecules

2. Main assertion
   - Solution spaces
   - Main assertion
   - Generalized Navier-Stokes equations
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Generalized nonlinear heat equation

We consider

$$\frac{\partial}{\partial t}u(x,t) + (-\Delta_x)^{\alpha}u(x,t) - Du^2(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T),$$

$$u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^n$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $u(x,t)$ a scalar function.

- $\alpha = 1$: classical nonlinear heat equation
- $\mathrm{D}f = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f$
- $\mathrm{D}u^2 \quad \rightsquigarrow \quad$ scalar model case of generalized Navier-Stokes equations

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Function spaces

Let $\phi_0 \in S(\mathbb{R}^n)$ with

$$\phi_0(x) = 1, \text{ if } |x| \leq 1 \quad \text{and} \quad \phi_0(x) = 0, \text{ if } |x| \geq \frac{3}{2}$$

and put

$$\phi_j(x) \quad = \quad \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}.$$

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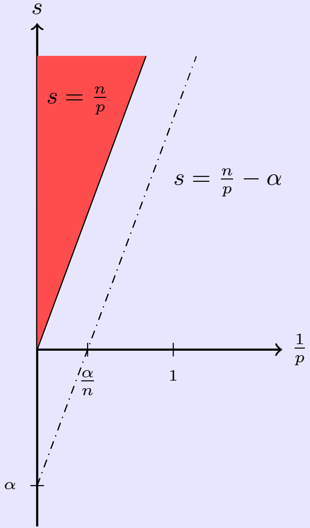
Definition:
Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, ($p < \infty$ for F-spaces), then

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}\phi_j \mathcal{F} f|L_p(\mathbb{R}^n)\|^q\right)^{1/q},$$

$$\|f|F^s_{p,q}(\mathbb{R}^n)\| = \|\left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}\phi_j \mathcal{F} f(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\|$$

with the usual modification if $q = \infty$.



- $u(\cdot,t) \in A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$, with s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra
- initial data $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$ with $s - \alpha < s_0 \leq s$

Procedure

Generalized nonlinear Heat equation:

$$\frac{\partial}{\partial t}u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T), \qquad (1)$$

$$u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^n. \qquad (2)$$

Solution as fixed point of the operator T_{u_0}

$$T_{u_0}u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T$$

in some weighted Lebesgue spaces $L_v((0,T),\, b,\, X)$.

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$$W_t^\alpha \omega = K_t^\alpha * \omega, \quad \omega \in S'(\mathbb{R}^n),$$

where

$$K_t^\alpha(x) = (2\pi)^{-n/2}\left(e^{-t|\xi|^{2\alpha}}\right)^\vee(x),\ x \in \mathbb{R}^n.$$

Methods

Transfer (1), (2) in a related fixed point problem based on the Duhamel formula

$$T_{u_0} u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha D u^2(x,\tau) \mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ t > 0.$$

Proposition 1 (B., Schmeißer):
Let $1 \leq p, q \leq \infty$ ($p < \infty$ for the F-spaces), $s \in \mathbb{R}$, $d \geq 0$ and $\alpha \in \mathbb{N}$. Then there is a constant $c > 0$, such that for all t with $0 < t \leq 1$ and $\omega \in A_{p,q}^s(\mathbb{R}^n)$

$$t^{d/2\alpha} \| W_t^\alpha \omega | A_{p,q}^{s+d}(\mathbb{R}^n) \| \leq c \, \| \omega | A_{p,q}^s(\mathbb{R}^n) \|.$$

Methods

$$T_{u_0}u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ t > 0.$$

- Decomposition of $A^s_{p,q}(\mathbb{R}^n)$ by means of wavelets, molecules.
- Proposition 1.
- Multiplication algebras:

$$A^s_{p,q}(\mathbb{R}^n) \cdot A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n)$$

  if $s > \frac{n}{p}$ and in some limiting cases $s = \frac{n}{p}$.
- Embeddings, lifts, duality.

wavelets: Let $\psi_F,\, \psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$, real-valued compactly supported Daubechies wavelets with $\widehat{\psi_F}(0) = (2\pi)^{-1}$ and

$$\int_{\mathbb{R}} x^v \psi_M(x)\mathrm{d}x = 0 \quad \text{for all } v \in \{0, \ldots, u-1\}.$$

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With $G^0 = \{F, M\}^n$ and $G^j = \{F, M\}^{n*} =: G^*,\ j \geq 1$ we define

$$\Psi^j_{G,m}(x) := 2^{jn/2} \prod_{l=1}^{n} \psi_{G_l}(2^j x_l - m_l) \quad G \in G^j,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n.$$

In particular $\Psi^j_{G,m}$ satisfies moment conditions if $G \in G^*$.

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u sufficiently large, $f \in S'(\mathbb{R}^n)$

$$f \in A^s_{p,q}(\mathbb{R}^n) \Longleftrightarrow f = \sum_{j,\,G,\,m} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in a^s_{p,q}(\mathbb{R}^n),$$

in particular $\|f\,|A^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda\,|a^s_{p,q}(\mathbb{R}^n)\|$.

molecules: L_∞-functions $b_{j,m} : \mathbb{R}^n \to \mathbb{C}$ with

$$|D^\gamma b_{j,m}(x)| \leq 2^{j|\gamma|}(1 + 2^j|x - 2^{-j}m|)^{-L}, \quad |\gamma| \leq K,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,$$

and

$$\int_{\mathbb{R}^n} x^\beta b_{j,m}(x)\mathrm{d}x = 0 \quad |\beta| < N,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n.$$

for some $K,\ N \in \mathbb{N}_0$ and $L > N + n - 1$.

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- similar decomposition, $f \in S'(\mathbb{R}^n)$

$$f \in A^s_{p,q}(\mathbb{R}^n) \Longleftrightarrow f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \mu^j_m b_{j,m}, \quad \mu \in \bar{a}^s_{p,q}(\mathbb{R}^n)$$

with $\|f|A^s_{p,q}(\mathbb{R}^n)\| \sim \inf \|\mu \mid \bar{a}^s_{p,q}(\mathbb{R}^n)\|$.

We know

$$\omega \in A^s_{p,q}(\mathbb{R}^n) \text{ then } \omega = \sum_{j,\,G,\,m} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}$$

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We want a similar decomposition for $W^\alpha_t \omega$ for fixed $t > 0$ where

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Split

$$\begin{aligned} W^\alpha_t \omega &= W^\alpha_t \omega^0_k + W^\alpha_t \omega_k \\ &= \sum_{j<k} \sum_{G,m} \lambda^{j,G}_m 2^{-jn} W^\alpha_t \Psi^j_{G,m} + \sum_{j \ge k} \sum_{G,m} \lambda^{j,G}_m \underbrace{2^{-jn} W^\alpha_t \Psi^j_{G,m}}_{b^j_{G,m}}. \end{aligned}$$

$b^j_{G,m}$ ... α-caloric wavelets

We show for $2^j t^{1/2\alpha} \geq 1$

$$b^j_{G,m}(x,t)_d = 2^{jd} t^{d/2\alpha} b^j_{G,m}(x,t), \quad j \geq k,\ G \in G^j,\ m \in \mathbb{Z}^n \tag{3}$$

are molecules in $A^{s+d}_{p,q}(\mathbb{R}^n)$ with $d \geq 0$ and

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Furthermore

$$\| W^\alpha_t \omega^0_k | A^{s+d}_{p,q}(\mathbb{R}^n) \| \leq c\, 2^{kd} \| \omega | A^s_{p,q}(\mathbb{R}^n) \|, \quad \text{for } t^{-\frac{d}{2\alpha}} \sim 2^{kd}.$$

$\curvearrowright$ Proposition 1.

Solution spaces

Solution spaces:

X Banach space with $X \subset S'(\mathbb{R}^n)$, $0 < T < \infty$, $b \in \mathbb{R}$, $1 \leq v \leq \infty$.

Then $L_v((0,T), b, X)$ contains all $f : (0,T) \to X$ such that

$$\|f|L_v((0,T), b, X)\| = \left(\int_0^T t^{bv} \|f(\cdot,t)|X\|^v \mathrm{d}t \right)^{1/v} < \infty$$

(with the usual modification if $v = \infty$).

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- restrictions below $\rightsquigarrow L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n)) \subset S'(\mathbb{R}^{n+1})$ regular distributions

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Mild solution: solutions u, which are a fixed point of T_{u_0} in a Banach space

Strong solution: unique mild solution, which belongs to $C([0,T), A^{s_0}_{p,q}(\mathbb{R}^n))$ for all initial data $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$

Theorem 1 (B., Schmeißer): Let $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p, q \leq \infty$ ($p < \infty$ for F-spaces) and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Let

$$0 < \lambda < g \leq 1, \quad \frac{2}{\alpha} < v \leq \infty, \quad a = \alpha - \frac{1}{v} - \alpha\lambda$$

and let $u_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$. Then there exists a number $T > 0$ such that

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= \quad 0 \text{ in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= \quad u_0(x) \text{ in } \mathbb{R}^n \end{aligned}$$

has a unique mild solution

$$u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)).$$

If, in addition, $p < \infty$, $q < \infty$ and

$$\frac{g}{2} \leq \lambda < g \leq 1 \text{ if } v < \infty \quad \text{and} \quad \frac{g}{2} < \lambda < g \leq 1 \text{ if } v = \infty$$

then the above solution is strong.

$\alpha = 1$: H. Triebel: Local Function Spaces, Heat and Navier-Stokes Equations

Stability and well-posedness

Local stability: For any $\varepsilon > 0$ there exists a $\delta > 0$ and a time $T > 0$ such that for all $0 < t < T$ holds

$$\|u_1(\cdot, t) - u_2(\cdot, t)|A^{s_0}_{p,q}(\mathbb{R}^n)\| \leq \varepsilon$$

if

$$\|u^1_0 - u^2_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \leq \delta$$

u_i solution of (1), (2) with corresponding initial data u^i_0, $i = 1, 2$.

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Well-posedness: A problem is called well-posed if there is a unique mild solution which is additionally strong and stable in the above sense.

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Well-posedness: A problem is called well-posed if there is a unique mild solution which is additionally strong and stable in the above sense.

Theorem (B., Schmeisser) Let u_i be strong solutions of (1), (2) with $u_0^i \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$ in the corresponding time interval $(0, T_i)$, $i = 1, 2$. Then

$$\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \leq c_0\|u_0^1 - u_0^2|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| + c_1\, t^{-\frac{g}{2}-\frac{a}{\alpha}+1}$$

for all $0 < t < T := \min(T_1, T_2)$.

Generalized Navier-Stokes equations

We consider

$$\begin{aligned} (\tfrac{\partial}{\partial t} + (-\Delta_x)^\alpha)\mathbf{u} + (\mathbf{u}, \nabla)\mathbf{u} + \nabla P &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in } \mathbb{R}^n \end{aligned}$$

in the reformulated version

$$\begin{aligned} (\tfrac{\partial}{\partial t} + (-\Delta_x)^\alpha)\mathbf{u} + \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in } \mathbb{R}^n. \end{aligned}$$

Here $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $0 < T \leq \infty$ and $\mathbb{P}$ denotes the Leray projector given by

$$(\mathbb{P}\, \mathbf{f}\,)^k = f^k + R_k \sum_{j=1}^{n} R_j f^j \quad k = 1, \ldots, n$$

where R_k stands for the scalar Riesz transform.

We prove an analogue result as for the generalized nonlinear heat equation using

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- $\mathbb{P}\,\nabla f = 0$ for any $f \in A^s_{p,q}(\mathbb{R}^n)_n$

Theorem (B., Schmeisser): Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $1 < p < \infty$, $1 \leq q \leq \infty$ and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Let

$$0 < \lambda < g \leq 1, \quad \frac{2}{\alpha} < v \leq \infty, \quad a = \alpha - \frac{1}{v} - \alpha\lambda$$

and let $\mathbf{u}_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n$. Then there exists a number $T > 0$ such that

$$\begin{aligned} (\frac{\partial}{\partial t} + (-\Delta_x)^\alpha)\mathbf{u}(x,t) + \mathbb{P}\,\mathrm{div}(\mathbf{u} \otimes \mathbf{u})(x,t) = 0, \qquad & \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad & \text{in } \mathbb{R}^n \end{aligned}$$

has a unique mild solution

$$\mathbf{u} \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n.$$

If, in addition, $q < \infty$ and

$$\frac{g}{2} \leq \lambda < g \leq 1 \text{ if } v < \infty \quad \text{and} \quad \frac{g}{2} < \lambda < g \leq 1 \text{ if } v = \infty$$

then the above solution is strong, that means $u \in C([0,T), A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n)$.

Thank you for your attention!